Chapter 5

Weak Duality Thm

Primal (LPP_p)

$$\max \ \vec{c}^T\vec{x}$$

$$FR_p \leftarrow \begin{cases} A\vec{x} \le \vec{b} \\ \vec{x} \ge 0 \end{cases}$$

Dual (LPP_d)

$$\min \ \vec{b}^T\vec{u}$$

$$FR_d \leftarrow \begin{cases} A^T\vec{u} \ge \vec{c} \\ \vec{u} \ge \vec{0} \end{cases}$$

(We don't assume $\vec{b} \ge \vec{0}$ or $\vec{c} \ge \vec{0}$)

Thm 5.2 $\quad \vec{x} \in FR_p, \ \vec{u} \in FR_d \ \Rightarrow \ \vec{c}^T\vec{x} \le \vec{b}^T\vec{u}$

Pf:
$$\begin{aligned} \vec{c}^T\vec{x} = \vec{x}^T\vec{c} &\le \vec{x}^T A^T \vec{u} \quad (\because A^T\vec{u} \ge \vec{c}, \ \vec{x} \ge \vec{0}) \\ &= \vec{u}^T A \vec{x} \\ &\le \vec{u}^T\vec{b} \quad (\because A\vec{x} \le \vec{b}, \ \vec{u} \ge \vec{0}) \ \# \end{aligned}$$

Corollary (Thm 5.3) $\quad \vec{x}_0 \in FR_p, \ \vec{u}_0 \in FR_d$ s.t. $\vec{c}^T\vec{x}_0 = \vec{b}^T\vec{u}_0$

$\Rightarrow \ \vec{x}_0$ is optimal for LPP_p & $\vec{u}_0$ is optimal for LPP_d

Pf: $\quad \vec{c}^T\vec{x}_0 = \vec{b}^T\vec{u}_0 \underset{\text{Thm 5.2}}{\ge} \vec{c}^T\vec{x} \quad \forall \vec{x} \in FR_p \quad \#$

Strong Duality Thm (Converse of Thm 5.3)

Thm 5.4 $\quad \vec{x}_0 \in FR_p, \ \vec{u}_0 \in FR_d$ s.t. $\vec{c}^T\vec{x}_0 = \vec{b}^T\vec{u}_0$

$\Longleftrightarrow \ \vec{x}_0$ is optimal for LPP_p OR $\vec{u}_0$ is optimal for LPP_d.

First let's prove the "OR" of the theorem is true

Pf: $\vec{x}_0$ is optimal for $LPP_p \underset{\text{Thm 5.4}}{\Rightarrow} \vec{c}^T\vec{x}_0 = \vec{b}^T\vec{u}_0 \underset{\text{Thm 5.3}}{\Rightarrow} \vec{u}_0$ is optimal for LPP_d.

\#

Thus we only need to prove: $\vec{x}_0$ is optimal for $LPP_p \Rightarrow \vec{c}^T\vec{x}_0 = \vec{b}^T\vec{u}_0$ where $\vec{u}_0$ is optimal for LPP_d.

pf: primal

$$\max \vec{c}^T\vec{x} \quad \left\{\begin{array}{l} A\vec{x}\le\vec{b} \\ \vec{x}\ge\vec{0}\end{array}\right. \Rightarrow \max \vec{c}^T\vec{x} + \underbrace{\vec{c}_s^T}_{=\vec{0}}\vec{x}_s \quad \left\{\begin{array}{l} A\vec{x}+I\vec{x}_s=\vec{b} \\ \vec{x},\vec{x}_s\ge 0\end{array}\right.$$

Since $\vec{x}_0$ is optimal for LPP_p $\Rightarrow$ $\vec{z}-\begin{pmatrix}\vec{c}\\ \vec{c}_s\end{pmatrix}\ge\vec{0}$ (i), $\vec{c}_s = 0$

Recall

(ii) $\vec{z}^T=\vec{c}_B^T\, Y$

(iii) $[A\,|\,I] = BY$

(iv) $\vec{b}=A\vec{x}_0=[B|R]\begin{bmatrix}\vec{x}_B\\0\end{bmatrix}=B\vec{x}_B \Rightarrow \vec{x}_B=B^{-1}\vec{b}$

(v) optimal value $=\vec{c}^T\vec{x}_0=(\vec{c}_B,\vec{c}_R)\begin{pmatrix}\vec{x}_B\\ \vec{0}\end{pmatrix}=\vec{c}_B^T\vec{x}_B$

$$\therefore \quad \vec{z}\overset{(ii)}{=}Y^T\vec{c}_B\overset{(iii)}{=}\begin{bmatrix}A^T\\ I\end{bmatrix}B^{-T}\vec{c}_B \qquad \left(B^{-T}=(B^{-1})^T=(B^T)^{-1}\right)$$

$$=\begin{bmatrix}A^TB^{-T}\vec{c}_B\\ B^{-T}\vec{c}_B\end{bmatrix}\underset{(i)}{\ge}\begin{pmatrix}\vec{c}\\ \vec{0}\end{pmatrix}$$

(vi) Define $\begin{pmatrix}\vec{z}_x\\ \vec{z}_{x_s}\end{pmatrix}=\begin{pmatrix}A^TB^{-T}\vec{c}_B\\ B^{-T}\vec{c}_B\end{pmatrix}$

$\Rightarrow$ (a) $A^T(B^{-T}\vec{c}_B)\ge\vec{c}$ (b) $B^{-T}\vec{c}_B\ge\vec{0}$

Thus let $\vec{u}_0=B^{-T}\vec{c}_B$ (vii) then $\left\{\begin{array}{l}A^T\vec{u}_0\ge\vec{c}\\ \vec{u}_0\ge 0\end{array}\right. \Rightarrow \vec{u}_0\in FR_d$

Moreover $\vec{b}^T\vec{u}_0=\vec{b}^TB^{-T}\vec{c}_B=\vec{c}_B^T\,B^{-1}\vec{b}\overset{(iv)}{=}\vec{c}_B^T\vec{x}_B\overset{(v)}{=}\vec{c}^T\vec{x}_0$ ✱

Corollary 1: The optimal solution of the dual problem, i.e. $\vec{u}_0$, is given as the reduced cost coefficient of the slack variables at the optimal tableau.

pf. From (vi), reduced cost coefficient of the slack variables

$=\vec{z}_s-\vec{c}_s^{\,=0}=\vec{z}_s$

$=B^{-T}\vec{c}_B \underset{(vii)}{=}\vec{u}_0$ ✱

<u>Corollary 2</u> The optimal values for the dual surplus variable $\vec{u}_s$ are given by the x_0 row underneath the primal structural variable $\vec{x}$.

pf: The dual problem is

$$\min \vec{b}^T\vec{u} \quad \begin{cases} A\vec{u} \geq \vec{c} \\ \vec{u} \geq \vec{0} \end{cases} \xrightarrow{\text{standardizing}} \max -\vec{b}^T u + \vec{0}^T\vec{u}_s \quad \begin{cases} A\vec{u} - I\vec{u}_s = \vec{c} \\ \vec{u}, \vec{u}_s \geq \vec{0} \end{cases}$$

At the optimal solution $(\vec{u}_0, \vec{u}_{0s}) \in FS$

$$A\vec{u}_0 - I\vec{u}_{0s} = \vec{c}$$

$$\Rightarrow \quad \vec{u}_{0s} = A\vec{u}_0 - \vec{c}$$

$$\overset{(vii)}{=} AB^{-T}\vec{c}_B - \vec{c} \qquad (viii)$$

$$\overset{(vi)}{=} (\vec{z}_{x_s} - \vec{c}) \ldots$$

Initial Tableau for Primal

$$\begin{array}{c|c:c|c|}
 & \vec{x} & \vec{x}_s & \vec{b} \\
\hline
 & A & I & \vec{b} \\
\hline
x_0 & -\vec{c}^T & \underset{0}{\underset{\|}{-\vec{c}_s^T}} & 0
\end{array}$$

Optimal Tableau for Primal

$$\begin{array}{c|c:c|c|l}
 & \vec{x} & \vec{x}_s & \vec{b} & \\
\hline
\vec{x}_B & \multicolumn{2}{c|}{Y = B^{-1}A \ (iii)} & B^{-1}\vec{b} = \vec{x}_B & (iv) \\
\hline
 & -(\vec{c}^T - \vec{z}_x^T) & -(\vec{c}_s - \vec{z}_{x_s}^T) & \vec{c}_B^T\vec{x}_B & (v)
\end{array}$$

$$\text{Corollary 2} \left\{ \begin{array}{c}
\| \ (vi) \\
-\vec{c}^T + A^T B^{-T}\vec{c}_B \\
\| \ (viii) \\
\vec{u}_{0s}
\end{array} \right. \quad \left. \begin{array}{c}
\| \ (vi) \\
-\vec{c}_s^T + B^{-T}\vec{c}_B \\
0 \quad \| \\
B^{-T}\vec{c}_B \\
\| \ (vii) \\
\vec{u}_0
\end{array} \right\} \text{Corollary 1}$$

Thm 5.7 At optimum $(\vec{X}_0, \vec{X}_{0s})$, $(\vec{u}_0, \vec{u}_{0s})$,

$$\vec{u}_0^T \vec{X}_{0s} = 0 \quad \text{and} \quad \vec{X}_0^T \vec{u}_{0s} = 0$$

pf: $\because$ $A^T \vec{u}_0 - I \vec{u}_{0s} = \vec{c}$

$$\Rightarrow \vec{X}_0^T A^T \vec{u}_0 - \vec{X}_0^T \vec{u}_{0s} = \vec{X}_0^T \vec{c} = \vec{c}^T \vec{X}_0 \quad (1)$$

$\because$ $A \vec{X}_0 + I \vec{X}_{0s} = \vec{b}$

$$\Rightarrow \vec{u}_0^T A \vec{X}_0 + \vec{u}_0^T \vec{X}_{0s} = \vec{u}_0^T \vec{b}$$

$$\Rightarrow \vec{X}_0^T A^T \vec{u}_0 + \vec{u}_0^T \vec{X}_{0s} = \vec{b}^T \vec{u}_0 \quad (2)$$

$\because$ optimality of $(\vec{X}_0, \vec{X}_{0s})$, $(\vec{u}_0, \vec{u}_{0s})$

$$\Rightarrow \vec{c}^T \vec{X}_0 = \vec{b}^T u_0$$

$\therefore$ (1)−(2) $\Rightarrow$ $\vec{u}_0^T \vec{X}_{0s} + \vec{X}_0^T \vec{u}_{0s} = \vec{0}$ (3)

$\because$ $\vec{u}_0, \vec{X}_{0s}, \vec{X}_0, \vec{u}_0 \geq \vec{0}$

(3) $\Rightarrow$ $\vec{u}_0^T \vec{X}_{0s} = 0$ & $\vec{X}_0^T \vec{u}_{0s} = 0$ # (4)

Thm 5.8 Converse of Thm 5.7 is true.

pf. If (4) is true $\Rightarrow$ (3) is true.

$$\Rightarrow \vec{b}^T \vec{u}_0 = \vec{X}_0^T A \vec{u}_0 + \vec{u}_0^T \vec{X}_{0s} \overset{(3)}{=} \vec{X}_0^T A^T \vec{u}_0 - \vec{X}_0^T \vec{u}_{0s} = \vec{c}^T X_0$$

By Strong Duality Thm. $\Rightarrow$ $\vec{u}_0$, $\vec{X}_0$ are optimal solutions. #

Eg 5.8 (iii)

initial tableau

	x_1	x_2	x_3	x_4	x_5	$\vec{b}$
x_4	1	3	4	1	0	30
x_5	1	4	-1	0	1	10
	-2	-7	3	0	0	0

optimal tableau

	x_1	x_2	x_3	x_4	x_5	$\vec{b}$
x_4	0	-1	5	1	-1	20
x_1	1	4	-1	0	1	10
x_0	0	1	1	0	2	20

$$B = \begin{array}{c} \begin{array}{cc} \vec{a}_4 & \vec{a}_1 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{array} \qquad B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Suppose

$$A = \begin{pmatrix} 1 & 3 & 4 & 1 & 0 \\ 1 & 4 & -1 & 0 & 1 \end{pmatrix} \longrightarrow \hat{A} = \begin{array}{c} \begin{array}{ccccc} & \hat{a}_2 & & & \end{array} \\ \begin{pmatrix} 1 & 1 & 4 & 1 & 0 \\ 1 & 3 & -1 & 0 & 1 \end{pmatrix} \end{array}$$

$$Y = \begin{pmatrix} 0 & -1 & 5 & 1 & -1 \\ 1 & 4 & -1 & 0 & 1 \end{pmatrix} \longrightarrow \hat{Y} = \begin{array}{c} \begin{array}{ccccc} & \hat{y}_2 & & & \end{array} \\ \begin{pmatrix} 1 & -2 & 4 & 1 & 0 \\ 1 & 3 & -1 & 0 & 1 \end{pmatrix} \end{array}$$

$$\left(\because \hat{A} = B\hat{Y} \Rightarrow \hat{a}_2 = B\hat{y}_2 \Rightarrow \hat{y}_2 = B^{-1}\hat{a}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right)$$

$$\begin{aligned} \vec{z} - \vec{c} &= \vec{c}_B^{\,T} Y - \vec{c} \\ &= (0,2)Y - \vec{c} \\ &= (2\ 8\ -2\ 0\ 0) - (2\ 7\ -3\ 0\ 0) \\ &= (0\ 1\ 1\ 0\ 0) \end{aligned} \longrightarrow \begin{aligned} \hat{z} - \vec{c} &= \vec{c}_B^{\,T} \hat{Y} - \vec{c} \\ &= (0,2)\hat{Y} - \vec{c} \\ &= (2\ 6\ -2\ 0\ 0) - (2\ 7\ -3\ 0\ 0) \\ &= (0, -1, 1, 0, 0) \end{aligned}$$

Thus the new tableau is

	x_1	x_2	x_3	x_4	x_5	$\vec{b}$
x_4	0	-2	5	1	-1	20
x_1	1	3*	-1	0	1	10
x_0	0	-1	1	0	2	20

After 1 iteration, we get the optimal tableau

	x_1	x_2	x_3	x_4	x_5	$\vec{b}$
x_4	2/3	0	13/3	1	-1/3	80/3
x_2	1/3	1	-1/3	0	1/3	10/3
x_0	1/3	0	2/3	0	7/3	70/3

#